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On L^1 Approximation of Discontinuous Functions

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I. Let f denote a real-valued Lebesgue integrable function defined on a nondegenerate compact real interval I. For n a fixed nonnegative integer, let P_n denote the set of real polynomials of degree n or less. If $q \in P_n$ is such that

$$\int_I |f - q| \, dx = \inf_{p \in P_n} \int_I |f - p| \, dx,$$

we call q a best approximant to f.

If f is continuous, q is known to be unique. In [1] Rivlin and Kripke show that q is unique when f has certain types of discontinuities. If, however, f has a jump discontinuity, q may or may not be unique. In this note we show that if f has a finite number of jump discontinuities and is continuous elsewhere, then f has a unique "distinguished" best approximant.

II. In order to prove our main theorem (Theorem 1), we make a definition and state two lemmas.

DEFINITION. We call x_0, an interior point of I, a *crossing* of a real-valued function f defined on I, if $f(x)(x - x_0)$ is either everywhere nonnegative or everywhere nonpositive in some neighborhood of x_0.

The following characterization lemma is a special case of [1, Theorem 1.3].

LEMMA 1. *Let $f \in L^1[I]$ and $q \in P_n$ be such that $f - q$ has only a finite number of zeros in I. Then*

$$\int_I |f - q| \, dx = \inf_{p \in P_n} \int_I |f - p| \, dx$$

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if and only if

$$\int_I p \operatorname{sgn}(f - q)\, dx = 0$$

for all $p \in P_n$.

LEMMA 2. *Let f be a real-valued Lebesgue integrable function on I, continuous there except possibly at $x_0 \in I$. Let $q \in P_n$ $(n \geq 1)$ be such that*

$$\int_I |f - q|\, dx = \inf_{p \in P_n} \int_I |f - p|\, dx.$$

Then $f - q$ has at least n zeros in $I - \{x_0\}$.

Proof of Lemma 2. Assume Lemma 2 is false. Then $f - q$ has exactly k zeros in $I - \{x_0\}$, $0 \leq k < n$. By Lemma 1,

$$\int_I 1 \cdot \operatorname{sgn}(f - q)\, dx = 0.$$

Thus $f - q$ has at least one crossing in I. Let $x_1 < x_2 < \cdots < x_m$ be the crossings of $f - q$. It follows that

$$\int_I \bar{p} \operatorname{sgn}(f - q)\, dx \neq 0$$

where $\bar{p}(x) = \prod_{i=1}^{m} (x - x_i)$. By Lemma 1, $m > n$. Since $f - q$ is continuous in I except possibly at x_0, $f - q$ must be zero at at least $m - 1$ of the x_i's each of which belongs to $I - \{x_0\}$. Thus $k \geq m - 1 \geq n$, which contradicts our assumption.

THEOREM 1. *Let f be a real-valued Lebesgue integrable function on I, continuous there except possibly at $x_0 \in I$. Let q and $\bar{q}$ belong to P_n $(n \geq 0)$ and be such that*

$$\int_I |f - q|\, dx = \int_I |f - \bar{q}|\, dx = \inf_{p \in P_n} \int_I |f - p|\, dx.$$

Then if q and $\bar{q}$ have the same leading coefficient, $q = \bar{q}$.

Proof. We assume $n \geq 1$ since the theorem is trivially true for $n = 0$. An application of the triangle inequality shows that

$$\int_I |f - \tfrac{1}{2}(q + \bar{q})|\, dx = \inf_{p \in P_n} \int_I |f - p|\, dx.$$

Thus

$$\int_I (|f - \tfrac{1}{2}(q + \bar{q})| - \tfrac{1}{2}|f - q| - \tfrac{1}{2}|f - \bar{q}|)\, dx = 0,$$

which implies that

$$|f - \tfrac{1}{2}(q + \bar{q})| = \tfrac{1}{2}|f - q| + \tfrac{1}{2}|f - \bar{q}|$$

on $I - \{x_0\}$. By Lemma 2 (applied to the polynomial $\frac{1}{2}(q + \bar{q})$), $f - \frac{1}{2}(q + \bar{q})$ has at least n zeros on $I - \{x_0\}$, which means that

$$\tfrac{1}{2}|f - q| + \tfrac{1}{2}|f - \bar{q}|$$

has at least n zeros on $I - \{x_0\}$. Thus $q = \bar{q}$ at at least n points of $I - \{x_0\}$, i.e., the polynomial $q - \bar{q}$, of degree $n - 1$ or less, has at least n zeros on I, which means that $q = \bar{q}$.

Remark. On the basis of Theorem 1, one may conjecture that if q and $\bar{q}$ are both best approximants to f such that the coefficients of x^k for some $k(0 \leqslant k < n)$ are the same in q and $\bar{q}$, then $q = \bar{q}$. The following example shows that, in general, this is not the case.

EXAMPLE. Let

$$f(x) = \begin{cases} 0 & (-1 \leqslant x \leqslant 2), \\ -1 & (2 < x \leqslant 3), \end{cases}$$

and let $n = 1$. For $-\frac{1}{3} \leqslant m < 0$ one has

$$\int_{-1}^{3} 1 \cdot \operatorname{sgn}(f - mx)\, dx = 0,$$

and

$$\int_{-1}^{3} x \cdot \operatorname{sgn}(f - mx)\, dx = 0.$$

Thus the linear polynomials $mx(-\frac{1}{3} \leqslant m < 0)$ are best approximants to f, and all of them have the same coefficient of x^{n-1}. (One notes that the zero polynomial is also a best approximant to f.)

III. In this section we state without proof extensions of Theorem 1.

By arguments similar to those above one can prove the following statements.

If f is a real-valued Lebesgue integrable function on I, continuous there except possibly throughout a set S consisting of exactly k distinct points ($k \geqslant 1$) of I, and if $q \in P_n$ ($n \geqslant k$) satisfies

$$\int_I |f - q|\, dx = \inf_{p \in P_n} \int_I |f - p|\, dx,$$

then $f - q$ has at least $n + 1 - k$ zeros on $I - S$.

Let f be a real valued Lebesgue integrable function on I, continuous there except possibly throughout a set consisting of exactly k distinct points ($k \geq 1$) of I, then if $q, \bar{q} \in P_n$ ($n \geq 0$) are such that

$$\int_I |f - q|\, dx = \int_I |f - \bar{q}|\, dx = \inf_{p \in P_n} \int_I |f - p|\, dx,$$

and if the k leading coefficients of q equal, respectively, the corresponding coefficients of $\bar{q}$, then $q = \bar{q}$.

If instead of using polynomials as the approximating functions one uses linear combinations of functions forming a Haar system, then one can prove an analog of Theorem 1. We recall that a sequence of continuous real-valued functions $f_1, ..., f_m$ defined on I is a Haar system if every nontrivial linear combination of $f_1, ..., f_m$ vanishes at at most $m - 1$ points of I.

THEOREM 2. *Let f be a real-valued Lebesgue integrable function on I, continuous there except possibly at $x_0 \in I$. Let $f_1, ..., f_m$ ($m \geq 2$) be a Haar system, and V the linear span of $f_1, ..., f_m$. Let w and $\bar{w}$ belong to V, and satisfy*

$$\int_I |f - w|\, dx = \int_I |f - \bar{w}|\, dx = \inf_{v \in V} \int_I |f - v|\, dx.$$

Further, suppose that for some i ($1 \leq i \leq m$), $f_1, ..., f_{i-1}, f_{i+1}, ..., f_m$ is also a Haar system. Then if w and $\bar{w}$ have the same coefficient of f_i, $w = \bar{w}$.

In proving Theorem 2 one uses the following fact: if $x_1, ..., x_k$ are distinct points in the interior of I ($1 \leq k \leq m - 1$), then there exists an element of V which changes sign at exactly these points.

Similar theorems hold in the event that f has k discontinuities, $1 < k \leq m - 1$.

REFERENCE

1. B. R. KRIPKE AND T. J. RIVLIN, Approximation in the Metric of $L^1(X, \mu)$, *Trans. Amer. Math. Soc.* **119** (1965), 101–122.